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Bakamjian–Thomas construction for quasistable states

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Abstract
As is evident from the multiple values listed in the particle data table for the mass and width of resonances, such as Z°, Δ and ρ, defining these resonance parameters uniquely and unambiguously remains an open problem. This problem is ultimately rooted in the absence of a state vector description of a resonance that has definite properties under spacetime transformations. We show that there exist irreducible representations of the causal Poincaré semigroup that provide such a state vector description to resonances, leading to well-defined mass and width parameters. Generated by an interaction-incorporating Poincaré algebra and characterized by the complex S-matrix pole position and spin of the resonance, these representations synthesize the Bakamjian–Thomas construction of relativistic interactions and the S-matrix description of resonances.

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1. Introduction

Currently, there is no consensus on the definitions of the mass and width of relativistic resonances. This ambiguity is starkly evident from the multiple mass and width values that the particle data table lists for certain resonances: two sets of values each for ρ and Δ and three sets of values for the Z°-boson [1]. These different values are obtained from fitting the same lineshape data from the same experiments and the differences are well outside experimental error. In contrast, mass is one of the fundamental parameters by which a stable elementary quantum system is defined. The notion has a wonderfully natural meaning as the eigenvalue of one of the two Casimir operators of an irreducible representation of the Poincaré group (where the other Casimir operator leads to the notion of spin). If resonances and decaying states are to be considered autonomous physical entities, as they are generally viewed by experimentalists and phenomenologists, then it is highly desirable to find a mathematical structure that allows their mass to emerge from an irreducible representation of a suitable set of relativistic spacetime transformations. Recall that well-defined values of mass and width are necessary not only for the purposes of the classification of resonances and decaying states as elementary particles; for instance, the mass of the Z° is a fundamental parameter of the standard model. Therefore, unambiguous, unique definitions of mass and width of resonances are of both theoretical and practical importance.

The conventional approach to relativistic resonances is based on local quantum field theory, the most popular framework for constructing physical theories that obey the principles of quantum mechanics and special relativity. The usual treatments based on quantum field theory do not permit resonances to be viewed as autonomous physical entities but rather as certain intermediaries [2]. As such, mass and width of resonances are defined not by appealing to the symmetry transformations of relativistic spacetime as done for stable particles. Instead, the singularity structures of the propagator are used to define them, as in the on-mass-shell definitions based on a renormalization point of the propagator.

The main difficulty of these approaches is the arbitrariness of definitions and the ambiguity of the resulting values. As first pointed out by Stuart [3], the on-mass-shell definitions of mass and width also suffer from the serious problem gauge non-invariance; for instance, on-shell mass of the Z°-boson is gauge-dependent at $O(g^4)$ and higher and its width at $O(g^6)$ and higher. The gauge dependence of these parameters can be removed by choosing the *S*-matrix pole as the renormalization point, and many of the subsequent attempts at defining the mass and width of relativistic resonances in fact use the *S*-matrix definition, see, e.g., [4]. However, these definitions all still carry a measure of arbitrariness and therewith a certain ambiguity remains in the mass and width values obtained from the lineshape data for resonances.

The point of view we advocate here is that it is mathematically tenable and physically desirable to treat quasistable states as autonomous entities along the same lines as stable particles. More precisely, the mass and width of a resonance should be defined from a theory that provides resonances with a state vector description that has well-defined properties under spacetime transformations. We also subscribe to the well-known fact that theories that synthesize the principles and quantum mechanics and special relativity need not necessarily assume the existence of quantum fields to mediate interactions. Our goal is to push on as far as possible with only the principles of quantum theory and special relativity. In this setting, by a relativistic quantum theory, we mean one in which there exists a unitary representation of the Poincaré group in the Hilbert space of the theory.

Following an earlier work by Dirac [5], Bakamjian and Thomas [6] constructed the first class of relativistic quantum theories of a directly interacting two-particle system. The key idea of the construction, which we will refer to as the BT-construction, is that interactions can be introduced as a perturbation to the invariant mass operator $M = M_0 + \Delta M$, akin to the non-relativistic case where the Hamiltonian absorbs the interactions, $H = H_0 + V$. Interaction incorporating self-adjoint operators that furnish a realization of the Poincaré algebra can be induced from this interacting mass operator M and the interaction-free generators of Poincaré transformations. Here, it is possible to choose a kinematic subalgebra that remains unaffected by the interactions, leading to different forms of dynamics [5]. Sokolov extended the construction from the two-particle case studied in [6] to an arbitrary number, conserved or not, in a manner that satisfies the cluster decomposition principle [7]. See [8] for the development of relevant ideas and [9] for an excellent comprehensive review of the subject and some interesting original results. In particular, many relativistic quantum systems and phenomena, such as bound states of quarks and nucleon–nucleon scattering, can be accommodated within the BT-construction [9]. However, the classification of these systems and phenomena in terms of whether they admit a BT-construction, a field theoretic construction or both remains an interesting open problem. In this paper, we will examine how the BT-construction may be extended to accommodate relativistic resonances and decaying states.

To illustrate the fundamental ideas, we will consider the scattering of two stable particles leading to the formation of a resonance. The main technical result we report is the existence of an irreducible representation of the causal Poincaré semigroup, the semidirect product of the Lorentz group and the semigroup of spacetime translations into the forward lightcone, that describes the resonance. This representation is characterized by the position of the S-matrix resonance pole and the spin-value of the partial wave in which the pole appears. Thus, a state vector description with well-defined transformation properties under symmetry operations appears possible for a resonance, and the spacetime translations of these state vectors constitute a fundamental criterion by which unique mass and width values of the resonance can be extracted from the pole position.

The causal Poincaré semigroup has a rather long history in connection with resonances and decaying states [10, 11]. Paper [11] is particularly important as it provides a complete classification of the representations of the Poincaré semigroup and identifies the ones that are suitable for describing resonances. These representations are identical to the ones we obtain here. In this regard, the main new contribution of the present work is that we explicate the dynamical content of these semigroup representations, i.e., as arising from the integration of a Poincaré algebra that incorporates interactions and characterized by the complex resonance pole of the S-matrix. We will primarily work in point-form dynamics for which the velocity basis with canonical spin is the natural basis. The velocity basis is made use of in [10]. The same basis is used also in [12] where the representations of the entire Poincaré group are used, as opposed to the semigroup, to describe resonances. The Poincaré semigroup also appears in another work [13] on resonances that uses the instant-form dynamics and quantum fields.

The organization of the paper is as follows. In sections 2 and 3, we review the partial wave analysis and the BT-construction of a two-particle system. In section 4, we construct the representation of the causal Poincaré semigroup when the two-particle system undergoes scattering leading to the formation of a resonance. Section 5 contains some concluding remarks.

2. Partial wave analysis

The Hilbert space of a stable elementary quantum system, often called a particle, furnishes a unitary, irreducible representation (UIR) of the Poincaré group $\mathcal{P}$ [14, 15]. Such a representation is characterized by the mass and spin of the particle as well as the sign of its energy, all of which are invariants under the action of the group. In addition to these kinematic parameters, the particle states may be characterized by invariant charges. Therefore, we denote the Hilbert space of the system by $\mathcal{H}^n(m, s)$, where m, s and n are the mass, spin and charge of the system, respectively. We always consider representations with positive energies.

The differential $\mathrm{d}U|_{(I,0)}$ of a unitary representation U of $\mathcal{P}$ furnishes a representation of the Poincaré Lie algebra by self-adjoint, unbounded operators. A convenient basis of this operator Lie algebra consists of the generators of spacetime translations P_μ and Lorentz transformations $J_{\mu\nu}$. These fulfil the commutation relations

$$\begin{aligned}
&[P_\mu, P_\nu] = 0,\\
&[P_\mu, J_{\rho\sigma}] = \mathrm{i}(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho),\\
&[J_{\mu\nu}, J_{\rho\sigma}] = \mathrm{i}(g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} + g_{\mu\sigma}J_{\nu\rho} - g_{\nu\sigma}J_{\mu\rho}).
\end{aligned} \tag{2.1}$$

The operators $M^2 = P_\mu P^\mu$ and $W = \frac{1}{M^2}\omega_\mu\omega^\mu$, where $\omega_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma}$, commute with the associative algebra generated by P_μ and $J_{\mu\nu}$. In a UIR, both are proportional to the identity, $M^2 = m^2 I$ and $W = s(s+1)I$.

The representation Hilbert space $\mathcal{H}^n(m, s)$ can always be realized as the space of L^2-functions defined on the Cartesian product of the spectra of a complete system of commuting operators (CSCO). Any four mutually commuting operators F_α from the algebra spanned by the generators P_μ and $J_{\mu\nu}$, along with the mass and spin operators M^2 and W can be chosen as a CSCO, and clearly there are infinitely many such equivalent CSCO. One common choice for F_α is the spatial momenta $\boldsymbol{P}$ and one component of the (canonical) spin operator S_3. The CSCO $\{\boldsymbol{P}, S_3, [M^2, W]\}$ does not have common eigenvectors that are bona fide elements of $\mathcal{H}^n(m, s)$, but such generalized eigenvectors $|\boldsymbol{p}, s_3, [m, s]n\rangle$ can be defined as functionals on a suitable dense subspace of $\mathcal{H}^n(m, s)$.

The UIR of the Poincaré group on $\mathcal{H}^n(m, s)$ is well defined by the action of the operators $U(\Lambda, a)$ on the generalized eigenvectors $|\boldsymbol{p}, s_3, [m, s]n\rangle$, given by

$$U(\Lambda, a)|\boldsymbol{p}, s_3[m, s]n\rangle = \mathrm{e}^{-\mathrm{i}a.\Lambda p} \sum_{s_3'} D^s_{s_3' s_3}(W(\Lambda, \Lambda p))|\boldsymbol{\Lambda p}, s_3'[m, s]n\rangle, \quad (2.2)$$

where $\boldsymbol{\Lambda p}$ is the spatial part of the four vector Λp and $W(\Lambda, p) = L^{-1}(p)\Lambda L(\Lambda^{-1}p)$ is a Wigner rotation.

The Hilbert space of two-particle states is the tensor product

$$\mathcal{H} = \mathcal{H}^{n_1}(m_1, s_1) \otimes \mathcal{H}^{n_2}(m_2, s_2). \quad (2.3)$$

The tensor product operators $U_0(\Lambda, a) = U_1(\Lambda, a) \otimes U_2(\Lambda, a)$, where $U_i(\Lambda, a)$ are defined by (2.2) for the particle of mass m_i and spin s_i, $i = 1, 2$, furnish a unitary representation of $\mathcal{P}$ in $\mathcal{H}$ (2.3). The Poincaré algebra that integrates to this tensor product representation is spanned by the sum of one-particle operators $P^{(i)}_\mu$ and $J^{(i)}_{\mu\nu}$, $i = 1, 2$:

$$\begin{aligned} P_{0\mu} &:= P^{(1)}_\mu \otimes I^{(2)} + I^{(1)} \otimes P^{(2)}_\mu, \\ J_{0\mu\nu} &:= J^{(1)}_{\mu\nu} \otimes I^{(2)} + I^{(1)} \otimes J^{(2)}_{\mu\nu}. \end{aligned} \quad (2.4)$$

The subscript 0 in (2.4) and in the representation U_0 indicates that there are no interactions between the two particles. Just as the one-particle operators (2.1), the operators (2.4) fulfil the commutation relations of the Poincaré algebra:

$$\begin{aligned} &[P_{0\mu}, P_{0\nu}] = 0, \\ &[P_{0\mu}, J_{0\rho\sigma}] = \mathrm{i}(g_{\mu\rho} P_{0\sigma} - g_{\mu\sigma} P_{0\rho}), \\ &[J_{0\mu\nu}, J_{0\rho\sigma}] = \mathrm{i}(g_{\nu\rho} J_{0\mu\sigma} - g_{\mu\rho} J_{0\nu\sigma} + g_{\mu\sigma} J_{0\nu\rho} - g_{\nu\sigma} J_{0\mu\rho}). \end{aligned} \quad (2.5)$$

The central elements are the square mass and spin operators,

$$M_0^2 = P_{0\mu} P_0^\mu; \qquad W_0 = \frac{1}{M_0^2} \omega_{0\mu} \omega_0^\mu, \quad (2.6)$$

where $\omega_{0\mu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P_0^\nu J_0^{\rho\sigma}$. Unlike generators (2.4), M_0 and W_0 are not the sums of the corresponding one-particle operators. The spectrum of the square mass operator M_0^2 is $[\mathsf{s}_0, \infty)$, where $\mathsf{s}_0 = (m_1 + m_2)^2$. The spin spectrum can be either $j = 0, 1, 2, \ldots$ if $|s_1 - s_2|$ is an integer or $j = 1/2, 3/2, 5/2, \ldots$ if $|s_1 - s_2|$ is a half-odd-integer.

It follows from these spectral values of M_0^2 and W that the tensor product representation $U_0 = U_1 \otimes U_2$ in (2.3) generated by the two-particle operators (2.4) is not irreducible. However, it can be reduced to a continuous direct sum of UIR's over s and j [16, 17]:

$$\mathcal{H}^{n_1}(m_1, s_1) \otimes \mathcal{H}^{n_2}(m_2, s_2) = \int_{\mathsf{s}_0}^\infty \mathrm{d}\mathsf{s} \sum_{j\eta} \mathcal{H}^\eta(\mathsf{s}, j). \quad (2.7)$$

Here, η is the degeneracy label that includes orbital angular momentum l, total spin s and species indices n. Each subspace $\mathcal{H}^\eta(\mathsf{s}, j)$ of (2.7) furnishes a UIR of $\mathcal{P}$ with mass $\sqrt{\mathsf{s}}$ and spin j. Just as done for the one-particle states, this irreducible representation can de defined by the transformation properties on the eigenvectors for a CSCO. If we choose $\{\boldsymbol{P}_0, S_{03}, [M_0, W_0]\}$ as the CSCO, then generalized eigevectors that transform irreducibly (for each fixed pair of values (s, j)) under U_0 are $|\boldsymbol{p}, j_3[\mathsf{s}, j]\eta\rangle$:

$$U_0(\Lambda, a)|\boldsymbol{p}, j_3[\mathsf{s}, j]\eta\rangle = \mathrm{e}^{-\mathrm{i}a.\Lambda p} \sum_{j_3'} D^j_{j_3' j_3}(W(\Lambda, \Lambda p))|\boldsymbol{\Lambda p}, j_3'[\mathsf{s}, j]\eta\rangle. \tag{2.8}$$

It is well known how these can be determined from the $|\boldsymbol{p}, s_{13}[m_1, s_1]\rangle \otimes |\boldsymbol{p}, s_{23}[m_2, s_2]\rangle$ by using the Clebsch–Gordan coefficients for the Poincaré group [16, 17].

3. The BT-construction

Let us now turn to the problem of introducing interactions into the above two-particle system along the lines of BT-construction. To that end, consider a perturbation of the mass operator (2.6) of the form

$$M := M_0 + \Delta M. \tag{3.1}$$

The central idea is to construct a set of ten operators P_μ and $J_{\mu\nu}$ such that the commutation relations (2.5) are fulfilled and the defining relation $M = P_\mu P^\mu$ holds for M defined by (3.1). The BT-construction shows [6, 9] how the interacting generators P_μ and $J_{\mu\nu}$ can be constructed from the free generators (2.4) and the interacting mass operator (3.1).

The first step of the construction is choosing a CSCO $\left\{F_\alpha, \left[M_0^2, W_0\right]\right\}$ for the two-particle states such that the operators F_α, which are functions of the free operators (2.4), all commute with the perturbation to the mass operator ΔM. The perturbation ΔM is also required to commute with the spin operator W_0. Thus,

$$[M, F_\alpha] = 0 \qquad \text{and} \qquad [M, W_0] = 0. \tag{3.2}$$

Therefore, the set of operators $\{F_\alpha, [M^2, W_0]\}$ is a CSCO for the interacting system. Now, we may invert the expressions F_α to obtain the interaction-free two-particle generators $P_{\mu 0}$ and $J_{\mu\nu 0}$. If the free mass operator (or functions thereof) appears in these inverted expressions, then we may replace it with the interaction-incorporating mass operator M to obtain the interacting generators. By virtue of (3.2), these fulfil the characteristic commutation relations (2.1) of the Poincaré algebra.

As a concrete example how this general construction goes, let us choose for the mutually commuting functions F_α the spatial components of the velocity operators defined by $Q_{0\mu} = \frac{P_{0\mu}}{M_0}$ and the third component of the canonical spin operator S_{03}, i.e., the CSCO is $\left\{\boldsymbol{Q}_0, S_{03}, \left[M_0^2, W_0\right]\right\}$. Thus, as in (3.2), the interaction term ΔM must satisfy the commutation relations

$$[\Delta M, \boldsymbol{Q}_0] = 0, \qquad [\Delta M, S_{03}] = 0. \tag{3.3}$$

Interacting operators can be defined by

$$\boldsymbol{P} = M\boldsymbol{Q}_0, \qquad H = MH_0 = M(I + \boldsymbol{Q}^2)^{1/2}, \qquad J_{\mu\nu} = J_{0\mu\nu}. \tag{3.4}$$

From (2.5) and (3.3), the operators P_μ and $J_{\mu\nu}$ defined by (3.4) fulfil the commutation relations of the Poincaré algebra, and in this sense, the interacting system preserves relativistic invariance. By construction, momentum operators are affected by interactions, while the Lorentz group generators are not. This choice amounts to what Dirac called the point-form dynamics [5]. It is evident from the defining equations (3.4) that the interaction term ΔM is invariant under Lorentz transformations. On the other hand, while it commutes with the velocity operators $Q_\mu = \frac{1}{M} P_\mu = Q_{0\mu}$, the interaction ΔM is not necessarily invariant under translations. These requirements imply that ΔM must be a function of only the internal variables, such as the magnitude of the relative momentum k and the degeneracy labels η (including l and s).

It follows from (3.4) that the spin operator $W = \frac{1}{M^2}\omega_\mu\omega^\mu$, where $\omega_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma}$, is identical to the free operator W_0 of (2.6). Therewith, we obtain a CSCO for the interacting system: $\{\boldsymbol{Q}, S_3, [M^2, W]\}$. From these considerations we see that the essential strength of the BT-construction is that it provides a way of inducing perturbations into all the generators of the ten-parameter group $\mathcal{P}$ from a perturbation to the single mass operator M_0. Therefore, it allows us to obtain a perturbation theory for the Poincaré group from the well-established perturbation theory for the one-dimensional Lie group $\mathbb{R}$.

Under the usual integrability conditions for operator Lie algebras [18], the interacting Lie algebra generated by (3.4) integrates to a unitary representation of $\mathcal{P}$ in (2.3). We denote this unitary representation by U, as opposed to U_0, the one generated by the free operators (2.4).

4. Resonance scattering and decay

Instead of directly integrating the interacting Poincaré algebra to obtain the unitary representation U of the Poincaré group, it is often more instructive to solve the eigenvalue problem for a CSCO and obtain a basis of generalized eigenvectors. This is a particularly relevant avenue to pursue when the interacting system undergoes resonance scattering. Thus, in anticipation of scattering, we will assume that ΔM is such that the spectrum of M is absolutely continuous and coincides with that of M_0. These assumptions are not very restrictive and generally made in non-relativistic scattering theory. Further, it is known from the non-relativistic theory that if the interaction $H - H_0$ satisfies certain regularity properties, then Møller operators exist and are asymptotically complete [19]. If the same properties are required of ΔM, then Møller operators exist and asymptotic completeness holds:

$$\Omega_\pm(M, M_0) = \lim_{\tau\to\mp\infty} \mathrm{e}^{\mathrm{i}M\tau}\,\mathrm{e}^{-\mathrm{i}M_0\tau}. \tag{4.1}$$

For notational economy, we will suppress the explicit reference to the mass operators and denote Møller operators simply by $\Omega_\pm$. It is evident from (4.1) that $\Omega_\pm$ satisfy the intertwining relations $M\Omega_\pm = \Omega_\pm M_0$ which, in turn, along with (3.4), imply

$$\begin{aligned} P_\mu\Omega_\pm &= \Omega_\pm P_{0\mu},\\ J_{\mu\nu}\Omega_\pm &= J_{0\mu\nu}\Omega_\pm = \Omega_\pm J_{0\mu\nu}. \end{aligned} \tag{4.2}$$

By asymptotic completeness, the operators $\Omega_\pm$ map the Hilbert space (2.3) unitarily onto itself such that, for $\varphi \in \mathcal{H}$, $\Omega_+\varphi := \phi^+$ are the scattering in-vectors and $\Omega_-\varphi := \psi^-$ are the out-vectors. When defined as elements of suitably defined dense subspaces, the in- and out-vectors ϕ^+ and ψ^-, respectively, may be expanded in terms of bases of generalized eigenvectors of a CSCO. While all CSCO are equivalent, for an interaction defining the point-form dynamics (3.4) the choice $\{\boldsymbol{Q}, S_3, [M^2, W]\}$ is most convenient. As will be shown below, this choice is particularly suitable for accommodating the analyticity properties of the mass wavefunctions suggested by the analytic structure of the S-matrix.

Generalized eigenvectors $|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta\rangle$ for the CSCO $\{\boldsymbol{Q}, S_3, [M^2, W]\}$ must be defined as elements of a space of functionals $\Phi^\times$ on a suitable dense subspace (of test functions) Φ of the Hilbert space $\mathcal{H}$. Such a space $\Phi^\times$ will contain the Hilbert space as a dense subspace. Therefore, we may extend the unitary operators $\Omega_\pm$ from $\mathcal{H}$ to operators $\Omega_\pm^\times$ in $\Phi^\times$. These extended operators $\Omega_\pm^\times$ map the generalized eigenvectors $|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta\rangle$ into interacting in- and out-generalized eigenvectors: $|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta^\pm\rangle = \Omega_\pm^\times|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta\rangle$. Like $|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta\rangle$, the $|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta^\pm\rangle$ must be properly defined as continuous antilinear functionals on suitable subspaces of the Hilbert space (2.3). We will shortly see that the analyticity properties of the S-matrix provide the clues as to how the relevant subspaces of $\mathcal{H}$ are to be constructed. This is in fact the point of departure of the formalism we develop from standard treatments of scattering and decay. A discussion of the lines of demarcation between the standard treatments and the rigged Hilbert space theory of resonance scattering can be found, for instance, in [20]. While this discussion is for the non-relativistic case, it easily translates into the relativistic case. When so defined as functionals, the vectors $|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta^+\rangle$ and $|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta^-\rangle$ furnish a basis for the in-vectors ϕ^+ and out-vectors ψ^-, respectively:

$$\begin{aligned}\phi^+ &= \sum_{jj_3\eta}\int_{\mathsf{s}_0}^{\infty}\mathrm{d}\mathsf{s}\int\frac{\mathrm{d}\boldsymbol{q}}{2q^0}|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta^+\rangle\langle^+\boldsymbol{q}, j_3[\mathsf{s}, j]\eta|\phi^+\rangle,\\ \psi^- &= \sum_{jj_3\eta}\int_{\mathsf{s}_0}^{\infty}\mathrm{d}\mathsf{s}\int\frac{\mathrm{d}\boldsymbol{q}}{2q^0}|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta^-\rangle\langle^-\boldsymbol{q}, j_3[\mathsf{s}, j]\eta|\psi^-\rangle.\end{aligned} \tag{4.3}$$

The vectors $|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta^\pm\rangle$ have the normalization $\langle\boldsymbol{q}', j_3'[\mathsf{s}', j']\eta'^\pm|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta^\pm\rangle = 2q^0\delta(\mathsf{s}'-\mathsf{s})\delta(\boldsymbol{q}'-\boldsymbol{q})\delta_{j_3'j_3}\delta_{j'j}\delta_{\eta'\eta}$. Thus, with (4.3), the S-matrix element $\langle\varphi|S\varphi\rangle = \langle\phi^+|\psi^-\rangle$ has the expansion

$$\langle\psi^-|\phi^+\rangle = \sum_{jj_3\eta}\int_{\mathsf{s}_0}^{\infty}\mathrm{d}\mathsf{s}\int\frac{\mathrm{d}\boldsymbol{q}}{2q^0}\langle\psi^-|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta^-\rangle S_j^\eta(\mathsf{s})\langle\boldsymbol{q}, j_3[\mathsf{s}, j]\eta^+|\phi^+\rangle, \tag{4.4}$$

where the reduced S-matrix $S_j^\eta(\mathsf{s})$ is defined (for elastic scattering) by

$$\langle\boldsymbol{q}', j_3'[\mathsf{s}', j']\eta'^-|\boldsymbol{q}, j_3[\mathsf{s}, j]\eta^+\rangle = \delta(\boldsymbol{q}'-\boldsymbol{q})\delta(\mathsf{s}'-\mathsf{s})\delta_{j'j}\delta_{j_3'j_3}\delta_{\eta'\eta}S_j^\eta(\mathsf{s}). \tag{4.5}$$

Aside from certain singularities, $S_j^\eta(\mathsf{s})$ is an analytic function defined on a multi-sheeted Riemann complex s-plane [21]. Resonances correspond to singularities of this function—in particular, a resonance is associated with a pair of simple poles of one partial wave of $S_j^\eta(\mathsf{s})$ defined by a particular pair of values (j, η). If the angular momentum value of the resonating partial wave is j_R, then j_R will be the spin value of the resonance. Since our focus here is such a resonance, we will only consider the value $j = j_R$ and suppress the summation over j and η in expansion (4.4). The pair of resonance poles of $S_{j_R}(\mathsf{s})$ occur at positions that are complex conjugates of each other, say $\mathsf{s} = \mathsf{s}_R$ and $\mathsf{s} = \mathsf{s}_R^*$ in the second Riemann sheet [21]. For the sake of definiteness, let us take s_R to be on the lower half plane, i.e., $\mathrm{Im}(\mathsf{s}_R) < 0$.

In order to bring forth the contribution of the resonance to $\langle\phi^+|\psi^-\rangle$, we must consider the extension of the integral over s in (4.4) into one defined over a contour that encircles the pole position $\mathsf{s} = \mathsf{s}_R$. This in turn requires that the integrand $\langle\psi^-|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-\rangle S_{j_R}(\mathsf{s})\langle\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^+|\phi^+\rangle$ has an analytic extension in s. Since $S_{j_R}(\mathsf{s})$ is already an analytic function, we only need to demand that the wavefunctions $\langle\psi^-|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-\rangle$ and $\langle\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^+|\phi^+\rangle$ have analytic extensions into the lower half complex s-plane, i.e., we must introduce boundary conditions into the wavefunctions in addition to the usual square integrability. For formation processes, the velocity wavefunctions $\langle\psi^-|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-\rangle$ and $\langle\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^+|\phi^+\rangle$ have the advantage that they can be analytically extended in s while keeping the velocity variables real. This leads to complex momenta of the form $\sqrt{\mathsf{s}}q_\mu$, where q_μ are real.

Recall that the Hilbert space for the unitary representation of the Poincaré group generated by interaction-incorporating operators (3.4) can always be realized as the space of L^2-functions defined on the Cartesian product of the spectra of any CSCO from the enveloping algebra of (3.4). For the CSCO $\{\boldsymbol{Q}, S_3, [M^2, W]\}$ and $W = j_R(j_R+1)I$, the L^2-realization of $\mathcal{H}_{j=j_R} = \int_{\mathsf{s}_0}^{\infty}\mathrm{d}\mathsf{s}\,\mathcal{H}(\mathsf{s}, j_R)$ is $L^2(\mathbb{R}_{\mathsf{s}_0}, \mathbb{R}^3)\otimes\mathbb{C}^{(2j_R+1)}$. Out of these L^2-functions, we must choose for $\langle\psi^-|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-\rangle$ and $\langle\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^+|\phi^+\rangle$ those which admit analytic extensions into the lower half complex plane. Further, for the contour deformation of the integral over s in (4.4) to be defined, these analytic extensions must decrease sufficiently fast for $|\mathsf{s}|\to\infty$. These requirements can be fulfilled if we choose $\langle\psi^-|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-\rangle$ and $\langle\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^+|\phi^+\rangle$

to be of Hardy class [22] from below (i.e., boundary values of analytic functions on the open lower half complex plane) in the square mass variable s. Specifically, let

$$\{\langle \boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-|\psi^-\rangle\} \equiv \mathcal{K}_+ := \mathcal{M}\cap\mathcal{H}^2_+\big|_{\mathbb{R}_{\mathsf{s}_0}} \otimes \mathcal{S}(\mathbb{R}^3)\otimes\mathbb{C}^{(2j_R+1)}, \qquad (4.6a)$$

$$\{\langle \boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^+|\phi^+\rangle\} \equiv \mathcal{K}_- := \mathcal{M}\cap\mathcal{H}^2_-\big|_{\mathbb{R}_{\mathsf{s}_0}} \otimes \mathcal{S}(\mathbb{R}^3)\otimes\mathbb{C}^{(2j_R+1)}. \qquad (4.6b)$$

Here, $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz space over $\mathbb{R}^3$, $\mathcal{H}^2_\pm$ are Hardy class functions on $\mathbb{C}^\pm$ and $\mathcal{M}$ is the subspace of Schwartz functions $\mathcal{S}(\mathbb{R})$ which, along with all of their derivatives, vanish at the origin. The symbol $|_{\mathbb{R}_{\mathsf{s}_0}}$ indicates the restrictions of the functions in $\mathcal{M}\cap\mathcal{H}^2_\pm$, the support of which is the whole real line, to the spectrum of M^2, $\mathbb{R}_{\mathsf{s}_0}$. Since $\langle \boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-|\psi^-\rangle \in \mathcal{H}^2_+$ implies $\overline{\langle \boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-|\psi^-\rangle} \in \mathcal{H}^2_-$, $\langle\psi^-|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-\rangle$ and therewith the integrand of (4.4) have the required analyticity properties on the lower half s-plane.

It is important to point out that requirements (4.6) do not impose additional restrictions on the interaction term ΔM. If, as required, $M = M_0 + \Delta M$ has an absolutely continuous spectrum bounded from below, then there exists a Hilbert space $L^2(\mathbb{R}_{\mathsf{s}_0}, \mathbb{R}^3)\otimes\mathbb{C}^{(2j_R+1)}$ in which M acts as a multiplication operator. Hardy function spaces $\mathcal{K}_\pm$ defined by (4.6) are dense subspaces of this Hilbert space. Furthermore, these spaces reduce the mass operator M and they can be equipped with a nuclear Frechét topology. Thus, denoting the topological duals of $\mathcal{K}_\pm$ by $\mathcal{K}^\times_\pm$, we have a pair of rigged Hilbert spaces [23]:

$$\mathcal{K}_\pm \subset L^2\big(\mathbb{R}_{\mathsf{s}_0}, \mathbb{R}^3\big)\otimes\mathbb{C}^{(2j_R+1)} \subset \mathcal{K}^\times_\pm. \qquad (4.7)$$

Definitions (4.6) imply that in- and out-vectors ϕ^+ and ψ^-, respectively, must belong to suitable dense subspaces of the Hilbert space $\mathcal{H}$ so that generalized eigenvectors $|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^\pm\rangle$ acting on these spaces as functionals yield the function spaces $\mathcal{K}_\pm$. Therefore, if we denote by $\Phi_\pm$ the set of vectors in $\mathcal{H}$ which have the L^2-realizations (4.6$\pm$), i.e., $\Phi_+ = \{\psi^-\}$ and $\Phi_- = \{\phi^+\}$, then we have $|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^\pm\rangle \in \Phi^\times_\mp$ and therewith the pair of abstract-rigged Hilbert spaces[1]:

$$\Phi_\pm \subset \mathcal{H} \subset \Phi^\times_\pm. \qquad (4.8)$$

With ϕ^+ and ψ^- well defined as the elements of $\Phi_\mp$ and $|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^\pm\rangle$ as the elements of $\Phi^\times_\mp$, the Dirac basis vector expansions (4.3) hold (without the sum over j and η) as rigorous mathematical identities (nuclear spectral theorem [24]).

Hardy functions are boundary values of functions analytic in the open (upper or lower) half complex plane. In other words, while the functions identified in spaces (4.6$\pm$) have real domains—as they should if they are subspaces of the Hilbert space $L^2(\mathbb{R}_{\mathsf{s}_0}, \mathbb{R}^3)\otimes\mathbb{C}^{(2j_R+1)}$—they have unique analytic extensions into the upper and lower complex s half-plane. Therefore, the choices (4.6$\mp$) as the spaces for in- and out-wavefunctions permit the deformation of integral (4.4) into a contour integral on the lower half complex s-plane. Since Hardy class functions vanish at infinity and since the integrand is analytic on the lower half s-plane except for the resonance pole at s_R, the integral of (4.4) over the square mass variable s can be written as the sum of an integral over $(-\infty, \mathsf{s}_0]$ and an integral over a contour that encircles s_R. This latter integral can be easily evaluated by the Cauchy theorem and Laurent expansion of $S_{j_R}(\mathsf{s})$ around $\mathsf{s} = \mathsf{s}_R$. Thus, we obtain

$$\begin{aligned}\langle\psi^-|\phi^+\rangle = c\sum_{j_{R3}}\int\frac{\mathrm{d}\boldsymbol{q}}{2q^0}\langle\psi^-|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle\langle\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^+|\phi^+\rangle \\ +\sum_{j_{R3}}\int_{-\infty}^{\mathsf{s}_0}\mathrm{d}\mathsf{s}\int\frac{\mathrm{d}\boldsymbol{q}}{2q^0}\langle\psi^-|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-\rangle S_{j_R}(\mathsf{s})\langle\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^+|\phi^+\rangle,\end{aligned} \qquad (4.9)$$

[1] The discrepancy between the $\pm$ for vector spaces and elements thereof is due to the way in- and out-states have been traditionally defined in physics and Hardy spaces from above and below have been defined in mathematics.

where c is a constant determined by the residue of the Laurent expansion. The vectors $|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle$ that appear in (4.9) are generalized eigenvectors of the CSCO $\{\boldsymbol{Q}, S_3[M^2, W]\}$ and they are well defined as elements of the dual space $\Phi_+^\times$. Since the momenta are defined as $P_\mu = MQ_\mu$, the vectors $|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle$ are also generalized eigenvectors of the momenta. Therefore,

$$\begin{aligned} M|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle &= \sqrt{\mathsf{s}_R}|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle, \\ P_\mu|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle &= \sqrt{\mathsf{s}_R}q_\mu|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle, \\ S_3|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle &= j_{R3}|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle, \\ W|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle &= j_R(j_R+1)|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle. \end{aligned} \tag{4.10}$$

These equalities show that the $|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle$ serve as a basis for a vector space in which the mass and spin operators are proportional to the identity, $M = \sqrt{\mathsf{s}_R}I$ and $W = j_R(j_R+1)I$, albeit the mass eigenvalue is now complex. Since parameters s_R and j_R (along with non-kinematic charges) are precisely the ones which define a resonance, the vector space spanned by $|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle$ furnishes a state vector description of the resonance.

As mentioned above, the Poincaré algebra spanned by the interaction-incorporating operators (3.4) integrates in the Hilbert space $\int_{\mathsf{s}_0}^\infty \mathrm{d}\mathsf{s}\,\mathcal{H}(\mathsf{s}, j_R)$ to a unitary representation of the Poincaré group. We denoted this representation by U, as opposed to the interaction-free representation U_0. In the L^2-realization of the Hilbert space, $L^2(\mathbb{R}_{\mathsf{s}_0}, \mathbb{R}^3) \otimes \mathbb{C}^{(2j_R+1)}$, the operators $U(\Lambda, a)$ can be defined to have the action

$$(U(\Lambda, a)f)(\mathsf{s}, \boldsymbol{q}, j_{R3}) = \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}}q.a} \sum_{j'_{R3}} D^{j_R}_{j_{R3}j'_{R3}}(W(\Lambda, q))f(\mathsf{s}, \boldsymbol{\Lambda}^{-1}\boldsymbol{q}, j'_{R3}). \tag{4.11}$$

If q is real, it follows from (4.11) and the defining properties of Hardy class functions that the subspace $\mathcal{K}_+$ of $L^2(\mathbb{R}_{\mathsf{s}_0}, \mathbb{R}^3) \otimes \mathbb{C}^{(2j_R+1)}$ remains invariant under $U(\Lambda, a)$ if and only if $a_0 \geqslant 0$ and $a^2 \geqslant 0$, i.e., if and only if a is a translation into the forward light cone. The set of elements

$$\mathcal{P}_+ := \{(\Lambda, a) : (\Lambda, a) \in \mathcal{P}; a^2 \geqslant 0, a_0 \geqslant 0\} \tag{4.12}$$

is a semigroup under the product rule of $\mathcal{P}$. We call $\mathcal{P}_+$ the *causal Poincaré semigroup*.

Since Φ_+ is isomorphic to (4.6*a*), we conclude that the restriction of the unitary representation U in $\mathcal{H}$ to Φ_+ furnishes a representation U_+ of the semigroup $\mathcal{P}_+$. This representation of $\mathcal{P}_+$ is differentiable in Φ_+ with respect to its nuclear Fréchet topology. By duality, then there exist a representation of $\mathcal{P}_+$ in $\Phi_+^\times$, differentiable with respect to its weak-* topology. The dual representation of a group is generally defined by the duality formula $\langle U^\times(g^{-1})F^+|\phi^+\rangle = \langle F^+|U(g)\phi^+\rangle$. However, for the Poincaré semigroup, we define it in the rigged Hilbert space (4.8)+) by $\left\langle U_+^\times(\Lambda^{-1}, \Lambda^{-1}a)F^-|\psi^-\right\rangle = \langle F^-|U_+(\Lambda, a)\psi^-\rangle$. Note that, unlike $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a)$, $(\Lambda^{-1}, \Lambda^{-1}a) \in \mathcal{P}_+$ when $(\Lambda, a) \in \mathcal{P}_+$. This property and the Abelian character of the translation semigroup ensures that $U_+^\times$ defined this way is a representation of $\mathcal{P}_+$ when U_+ is one. Then, generalized eigenvectors $|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-\rangle$ transform as

$$U_+^\times(\Lambda, a)|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^-\rangle = \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}}a.\Lambda q} \sum_{j'_{R3}} D^{j_R}_{j'_{R3}j_{R3}}(W(\Lambda, \Lambda q))|\boldsymbol{\Lambda}q, j'_{R3}[\mathsf{s}, j_R]^-\rangle. \tag{4.13}$$

Similarly, there exists a representation of $\mathcal{P}_+$ in Φ_- which is defined by the transformation rule of the basis vectors $|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^+\rangle$. For $(\Lambda, a) \in \mathcal{P}_+$,

$$U_-^\times(\Lambda, a)|\boldsymbol{q}, j_{R3}[\mathsf{s}, j_R]^+\rangle = \mathrm{e}^{\mathrm{i}\sqrt{\mathsf{s}}a.\Lambda q} \sum_{j'_{R3}} D^{j_R}_{j'_{R3}j_{R3}}(W(\Lambda, \Lambda q))|\boldsymbol{\Lambda}q, j'_{R3}[\mathsf{s}, j_R]^+\rangle. \tag{4.14}$$

As elements of $\Phi_+^\times$, resonance basis vectors also transform as (4.13):

$$U_+^\times(\Lambda, a)|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle = \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}_R}a.\Lambda q}\sum_{j'_{R3}} D^{j_R}_{j'_{R3}j_{R3}}(W(\Lambda, \Lambda q))|\boldsymbol{\Lambda q}, j'_{R3}[\mathsf{s}_R, j_R]^-\rangle. \quad (4.15)$$

The transformation formula (4.15) defines an irreducible representation of the causal Poincaré semigroup in the space of vectors spanned by the resonance basis vectors, $\sum_{j_{R3}}\int\frac{\mathrm{d}q}{2q^0}|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle\varphi(\boldsymbol{q}, j_{R3})$. In this vector space, the mass and spin operators act as in (4.10), $M = \sqrt{\mathsf{s}_R}I$ and $W = j_R(j_R+1)I$. Therefore, in the same vein as stable elementary quantum systems have representation as the UIR's of $\mathcal{P}$, resonance states have representation as the irreducible representations of $\mathcal{P}_+$ characterized by the complex square mass value s_R and half-odd-integer spin value j_R.

Furthermore, the transformation properties of $|\boldsymbol{q}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle$ under $\mathcal{P}_+$ provide a unique, unambiguous criterion for extracting the resonance mass and width values from the pole position, the question that motivated this study. To that end, let us consider the time evolution of the rest state. It follows from the general transformation formula (4.15):

$$U_+^\times(I, t)|\boldsymbol{0}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle = \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}_R}t}|\boldsymbol{0}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle. \quad (4.16)$$

This shows that the amplitudes $|\langle\psi^-(t)|\boldsymbol{0}, j_{R3}[\mathsf{s}_R, j_R]^-\rangle|^2$ decay as $\mathrm{e}^{-\Gamma t}$, where $\frac{\Gamma}{2} = \mathrm{Im}(\sqrt{\mathsf{s}_R})$, the half-width of the Lorentzian mass distribution characterized by the pole position s_R. Therefore, the representations of $\mathcal{P}_+$ synthesize the resonance and decaying state aspects quasistable states and establishes the relation $\tau = \frac{1}{\Gamma}$ between the lifetime of a decaying state and the width of a resonance as an exact identity. Since only the definition $\Gamma = 2\,\mathrm{Im}(\sqrt{\mathsf{s}_R})$ of width fulfils this identity, we have a unique definition of the resonance width as $\Gamma = 2\,\mathrm{Im}(\sqrt{\mathsf{s}_R})$ and therewith the resonance mass as $m = \mathrm{Re}(\sqrt{\mathsf{s}_R})$ such that $\mathsf{s}_R = (m - \mathrm{i}\Gamma/2)^2$. When the lineshape data for the Z^0-boson are fitted using these definitions, the resulting mass and width values are different from those obtained by the on-mass-shell scheme and other techniques [25, 4].

5. Conclusion

We have shown that the BT-construction [6, 9] can be extended to describe resonance states formed in two-particle scattering. The key elements of the construction are the inclusion of interactions in the mass operator (3.1) and the use of the velocity basis and Hardy class functions (4.6). From these considerations, we deduce that there exists an irreducible representation of the causal Poincaré semigroup uniquely characterized by the resonance pole position s_R and the spin value j_R of the resonating partial S-matrix, $S_{j_R}(\mathsf{s})$. These representations tie together resonances and decaying states into a single physical entity leading to unique, unambiguous definitions of resonance mass and width.

The particular details of a given resonance scattering process can be, in principle, encoded in the mass perturbation term ΔM as a function of the internal degeneracy labels. The formalism presented here is independent of such model mass operators and instead uses the description of a resonance as a pole of the S-matrix. As such, it can accommodate all model mass operators that give rise to the same pole structure of the S-matrix.

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References

[1] Eidelman S *et al (Particle Data Group)* 2004 *Phys. Lett.* B **592** 1
[2] Peskin M and Schroeder D 1995 *An Introduction to Quantum Field Theory* (Reading, MA: Addison-Wesley)
[3] Stuart R G 1991 *Phys. Lett.* B **262** 113
[4] For a review of the problems with and different approaches to defining mass and width of a resonance, see Bohm A and Sato Y 2005 *Phys. Rev.* D **71** 085018 (and references therein)
[5] Dirac P A M 1949 *Rev. Mod. Phys.* **21** 392
[6] Bakamjian B and Thomas L H 1953 *Phys. Rev.* **92** 1300
[7] Sokolov S N 1977 *Dokl. Akad. Nauk. USSR* **233** 575
[8] Foldy L 1961 *Phys. Rev.* **122** 275
Coester F and Polyzou W N 1982 *Phys. Rev.* D **26** 1348
[9] Keister B and Polyzou W N 1991 *Advances in Nuclear Physics* vol 20 (New York: Plenum)
[10] Zwanziger D 1963 *Phys. Rev.* **131** 2819
[11] Schulman L S 1970 *Ann. Phys.* **59** 201
[12] Simonius M 1970 *Helv. Phys. Acta* **43** 223
[13] Antoniou I, Gadella M, Prigogine I and Pronko G P 1998 *J. Math. Phys.* **39** 2995
[14] Wigner E P 1939 *Ann. Math.* **40** 149
[15] Weinberg S 1995 *Quantum Theory of Fields* vol I (Cambridge: Cambridge University Press)
[16] Macfarlane A J 1962 *Rev. Mod. Phys.* **34** 41
Joos H *Fortschr. Phys.* **10** 651962
[17] Bohm A and Kaldass H 1999 *Phys. Rev.* A **60** 4606
[18] Nelson E 1959 *Ann. Math.* **70** 572
Flato M *et al* 1972 *Ann. Sci. Ec. Norm. Sup.* **4** 423
Wickramasekara S and Bohm A 2002 *J. Phys. A: Math. Gen.* **35** 805
[19] Reed M and Simon B 1979 *Scattering Theory* (New York: Academic)
[20] Bohm A and Gadella M 1989 *Dirac Kets, Gamow Vectors and Gelfand Triplets (Springer Lecture Notes in Physics* vol 348) (New York: Springer)
[21] Eden R J, Landshoff P V, Olive P J and Polkinghorne J C 1962 *The Analytic S-Matrix* (Cambridge: Cambridge University Press)
[22] Koosis P 1980 *Introduction to $\mathcal{H}^p$ Spaces (London Math. Soc. Lecture Notes* vol 40) (Cambridge: Cambridge University Press)
[23] Wickramasekara S Relativistic dynamics of quasistable states: I. A perturbation theory for the Poincaré group *Preprint* Grinnell College
Wickramasekara S Relativistic dynamics of quasistable states: II. Differentiable representations of the causal Poincaré semigroup *Preprint* Grinnell College
[24] Gel'fand I M and Vilenkin N Ya 1964 *Generalized Functions* vol 4 (New York: Academic)
[25] Bohm A and Harshman N L 2000 *Nucl. Phys.* B **581** 91